Assignment 3: Line Integrals and Vector Fields

Be sure to show all work, not just the final answer. The assignment is due at the beginning of class, August 11th.

1. Evaluate the following integrals:

   (a)
   $$\int_C x^3 + y \, ds$$
   where C is given by $(x(t), y(t)) = (3t, t^3)$, $0 \leq t \leq 1$.

   (b)
   $$\int_C x^2 + y^2 \, dx$$
   where C is the line segment from $(1, 4)$ to $(-2, 8)$.

   (c) $\int_C \mathbf{F} \cdot dr$, where $\mathbf{F}(x, y, z) = xz\mathbf{i} + (y + z)\mathbf{j} + x\mathbf{k}$, and C is given by the parametric equation (e^t, e^{-t}, e^{2t}), $-1 \leq t \leq 1$.

2. For the vector field $\mathbf{F}(x, y) = (xy)\mathbf{i} + (y^2)\mathbf{j}$,

   (a) Sketch the vector field.

   (b) Evaluate $\int_C \mathbf{F} \cdot dr$, where C is the line segment from $(-2, 0)$ to $(2, 0)$, followed by the bottom half of the circle $x^2 + y^2 = 4$.

3. For the vector field $\mathbf{F}(x, y) = (2x + y^3)\mathbf{i} + (3xy^2 + 4)\mathbf{j}$,

   (a) Show that $\mathbf{F}$ is conservative.

   (b) Find f such that $\Delta f = \mathbf{F}$.

4. For the vector field $\mathbf{F} = (8xz)\mathbf{i} + (1 - 6yz^3)\mathbf{j} + (4x^2 - 9y^2z^2)\mathbf{k}$,

   (a) Find f such that $\Delta f = \mathbf{F}$.

   (b) Evaluate $\int_C \mathbf{F} \cdot dr$, where C is the line from $(0, 2, 3)$ to $(4, 2, 5)$, followed by the line from $(4, 2, 5)$ to $(4, 0, 1)$, followed by the line from $(4, 0, 1)$ to $(0, 2, 3)$.

** Extra practice questions: you do not need to hand these in.

5. Evaluate
$$\int_C (e^{x^2} + y)\,dx + (x^2 + \tan\sqrt{y})\,dy$$
where C is the line from $(a, 0)$ to $(0, a)$, followed by the line from $(0, a)$ to $(0, 0)$, followed by the line from $(0, 0)$ to $(a, 0)$.

6. As a space shuttle descends from its initial position in orbit to its landing spot in the Pacific Ocean, its position (relative to the earth's centre) is given by the parametric equation
$$p(t) = \left(200t, \sin\left(\frac{2\pi}{5}t\right), 10^4 \cos\left(\frac{\pi}{10}t\right)\right)$$
where t is the time in hours after the shuttle begins its descent. The shuttle takes 5 hours to descend. As it descends, the crew of the shuttle needs to determine how much work gravity is doing on the shuttle. Calculate the total work that is being done by gravity, if the force of gravity $\mathbf{F}$ on the shuttle is given by Δf, where
$$f = \frac{10^5 G}{\sqrt{x^2 + y^2 + z^2}}$$
(G is a constant).